MASTER'S COMPREHENSIVE EXAM IN
Math 600 -REAL ANALYSIS
January 2020

Do any three (out of the five) problems. Show all work. Each problem is worth ten points.

Q1 (a) Suppose (M, d) is a metric space where every open set is closed. Show that every real-valued function on M is continuous. [HINT: Consider single element sets.]

(b) Let (M, d) be a metric space and suppose that every real-valued function on M is continuous. Show that every open set in M is closed. [HINT: For every $x \in M$, consider the function $h_x : M \to \mathbb{R}$ defined by $h_x(y) = 1$ if $y = x$ and $h_x(y) = 0$ if $y \neq x$.]

Q2 In a metric space, provide the definitions of connected and arcwise (pathwise) connected sets. How are they related?

(a) Are $\mathbb{R}^n$ and $\mathbb{R}^n \setminus \{0\}$ connected? arcwise connected? For your answers, provide simple justifications.

(b) On $\mathbb{R}^2$, consider the function $f(x, y) = (x^2 + y^2)\, e^{\sin(x^2+y^2)}$. Show that the range of f is $[0, \infty)$.

(c) On $\mathbb{R}^2 \setminus \{0\}$, consider the function $g(x, y) = \frac{e^{\sin(x^2+y^2)}}{x^2+y^2}$. Show that the range of g is $(0, \infty)$.

Q3 You are given the series

$$\sum_{n=1}^{\infty} \exp\left(\frac{x^2}{n} - nx\right),$$

where $x \in (0, \infty)$.

(a) Show that the series converges uniformly on $[a, b]$ for all $a, b > 0$ with $a < b$.

(b) Explain why the sum is well defined and continuous on $(0, \infty)$.

(c) Show that the series does not converge uniformly on $(0, b]$ for any $b > 0$.

(d) Show that the series does not converge uniformly on $[a, \infty)$ for any $a > 0$.

Q4 Consider the space $C[0, 1]$ consisting of continuous functions $f : [0, 1] \to \mathbb{R}$, equipped with the sup-norm.

(a) State necessary and sufficient conditions for a subset $S \subset C[0, 1]$ to be compact.

(b) Let $A \subset C[0, 1]$ be defined by

$$A = \{f \in C[0, 1] \,|\, f(0) = 0,\ |f(x) - f(y)| \leq |x - y|\, \forall x, y \in [0, 1]\}.$$

Show that A is compact.

(c) Let $B \subset C[0, 1]$ be defined by

$$B = \{f \in C[0, 1] \,|\, 0 \leq f(x) \leq 1\, \forall x \in [0, 1]\}.$$

Show that B is closed and bounded, but that B is not compact.

Q5 (a) Provide the definition of the Fréchet derivative of a map $F : V_1 \to V_2$ where $(V_i, \|.\|_i)$ are finite dimensional normed vector spaces.

(b) Let $f, g : \mathbb{R} \to \mathbb{R}$ satisfy $f(0) = 0$, $g(0) = \lambda$, f is differentiable at 0, $f'(0) = \mu$ and g is continuous at 0.

Let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined by $F(x, y) = f(x)g(y)$ for all $(x, y) \in \mathbb{R}^2$.

Show that both partials $\frac{\partial F}{\partial x}(0, 0)$ and $\frac{\partial F}{\partial y}(0, 0)$ exist by computing.

Prove that F has directional derivatives along all directions at $(0, 0)$.

Prove that F is Fréchet differentiable at $(0, 0)$.