MASTER'S COMPREHENSIVE EXAM IN
Math 603-MATRIX ANALYSIS
August 2019

Do any three (out of the five) problems. Show all work. Each problem is worth ten points.

Q1 Let $A \in \mathbb{R}^{n\times n}$ be a skew symmetric matrix, i.e., $A^T = -A$.

(a) Show that if n is odd, then $\det(A) = 0$.

(b) Show that each eigenvalue of A is either zero or pure imaginary (i.e., of the form $\imath a$ for a real number $a \neq 0$).

(c) Show that $x^T Ax = 0$ for any $x \in \mathbb{R}^n$.

(d) Let $D \in \mathbb{R}^{n\times n}$ be a positive definite matrix. Use (c) to show $D + A$ is invertible.

Q2 Let $\mathbb{R}^{2\times 2}$ be the space of 2×2 real matrices.

(a) Show that $\mathcal{X} := \{A = [a_{ij}] \in \mathbb{R}^{2\times 2} \,|\, a_{11} = a_{22} = 0\}$ is a subspace of $\mathbb{R}^{2\times 2}$. Find a basis for $\mathcal{X}$ (without proof), and determine the dimension of $\mathcal{X}$.

(b) Consider the standard inner product $\langle A, B\rangle := \text{trace}(A^T B)$ on $\mathbb{R}^{2\times 2}$. Find a basis for $\mathcal{X}^\perp$, and prove that your finding is indeed a basis for $\mathcal{X}^\perp$.

(c) Let $\{A, B, C, D\}$ be a basis for $\mathbb{R}^{2\times 2}$. Show that $\mathbb{R}^{2\times 2}$ is the direct sum of $\mathcal{Y} := \text{span}\{A+B, B\}$ and $\mathcal{Z} := \text{span}\{B-C, C+D\}$. Find the projection of $A+2B+3C+4D$ onto $\mathcal{Y}$.

Q3 (a) Show that $A \in \mathbb{R}^{n\times n}$ has rank one if and only if there exist two nonzero (column) vectors $x, y \in \mathbb{R}^n$ such that $A = xy^T$.

(b) For the rank-one matrix A in (a), determine a necessary and sufficient condition for its trace to be zero in terms of x and y, and explain why.

(c) Let $A = xy^T$ and $B = zw^T$ be rank-one matrices for nonzero vectors x, y, z and w in $\mathbb{R}^n$. What is the largest possible rank of $A + B$? Show that this largest possible rank is achieved if and only if both $\{x, z\}$ and $\{y, w\}$ are linearly independent sets.

Q4 Define a unitary matrix, and state when two matrices A and B are unitarily similar. Show:

(a) If $A = A^*$ and A is unitarily similar to B, then $B = B^*$.

(b) If a Hermitian matrix A is positive definite and B is unitarily similar to A, then B is also positive definite.

(c) The eigenvalues of a unitary matrix satisfy $\bar{\lambda} = \lambda^{-1}$.

(d) If A is normal and B is unitarily similar to A, then B is also normal.

Q5 Let V be an n-dimensional inner product space over $\mathbb{C}$.

(a) Let T be a self-adjoint (or Hermitian matrix) operator on V, that is

$$\langle u, Tv\rangle = \langle Tu, v\rangle, \qquad \forall u, v \in V. \tag{1}$$

Show that T is a nonnegative self-adjoint operator on V (that is $\langle v, Tv\rangle \geq 0$ for $v \in V$) if and only if the eigenvalues of T are all nonnegative real numbers.

(b) Let T_1 and T_2 be two self-adjoint operators on V in the sense of (1). Prove or reject: $T_1 T_2 + T_2 T_1$ is also self-adjoint.

(c) Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Show that $U^\perp$ is invariant under T^* if U is invariant under T.