MASTER'S COMPREHENSIVE EXAM IN
Math 603-MATRIX ANALYSIS
January 2018

Solve any **three** *problems. Show all work. Each problem is worth ten points.*

Notation and definitions:

A square matrix A with complex entries is called normal if $A^*A = AA^*$, where $A^* = \overline{A^T}$ (complex conjugate of transpose). A square matrix A is called nilpotent if there exists $k \in \mathbb{N}$ so that $A^k = 0$. For any $m \times n$ matrix, $N(A)$ denotes the nullspace of A and $R(A)$ the range of A.

Q1 (a) Let x be a nonzero (column) vector in $\mathbb{R}^n$. Show that the $n \times n$ matrix $I - \frac{xx^T}{\|x\|_2^2}$ is positive semi-definite, and determine its null space, rank, and all of its eigenvalues, and justify your answers.

(b) Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$. Let $r = rank(B)$ and $s = dim(N(A) \cap R(B))$. Find values of r and s so that the columns of AB are linearly independent. Moreover, show that the values you found are the only ones for which columns of AB are linearly independent.

Q2 (a) Show that if A is normal, then $N(A) \perp R(A)$ and $\mathbb{C}^n = N(A) \oplus R(A)$. (Do not simply state that this is a known result, prove it!)

(b) Assume A has the block structure $A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$ with B being a $k \times k$ matrix. Show that A is normal if and only if B and C are normal.

(c) Show that a triangular matrix is normal if and only if it is diagonal. (For less credit you may assume $n = 2, 3$).

Q3 Let A be an $m \times n$ real matrix so that A^TA is invertible and let $P = A(A^TA)^{-1}A^T$.

(a) Show that the matrix P is the orthogonal projector onto $R(A)$.

(b) Let B be an $m \times p$ real matrix so that B^TB is invertible and $R(B) \cap R(A) = \{0\}$. Show that $B^T(I - P)B$ is invertible.
(Hint: First show $B^T(I - P)B = (QB)^T(QB)$, where $Q = I - P$.)

Q4 Let V be an inner product space over $\mathbb{C}$, with inner product $\langle u, v\rangle$.

(a) Prove that any finite set S of nonzero, pairwise orthogonal vectors is linearly independent.

(b) If $T : V \to V$ is a linear operator satisfying $\langle T(u), v\rangle = \langle u, T(v)\rangle$ for all $u, v \in V$, prove that all eigenvalues of T are real.

(c) If $T : V \to V$ is a linear operator satisfying $\langle T(u), v\rangle = \langle u, T(v)\rangle$ for all $u, v \in V$, prove that the eigenvectors of T associated with distinct eigenvalues λ and μ are orthogonal.

Q5 Given the 2-by-2 real matrix $A = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}$, determine the set of all real a, b such that A is

(a) orthogonal.

(b) symmetric, positive definite.

(c) nilpotent.

(d) normal.