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MASTER'S COMPREHENSIVE EXAM IN
Math 603 - MATRIX ANALYSIS
August 2018

Do any **THREE** problems. **Show all your work.** Each problem is worth 10 points.

Q1 Let $V = C[-1,1]$ be the vector space of continuous functions from $[-1,1]$ to R.

(a) Define $\langle \cdot, \cdot \rangle : V \times V \to R$ by

$$\langle f, g \rangle = \int_{-1}^{1} f(\xi) g(\xi) \frac{d\xi}{\sqrt{1-\xi^2}}.$$

Show that this is an inner product on V.

(b) The *Tchebychev polynomials*, $\{T_n(x)\}_{n=1}^{\infty}$, are defined by declaring T_n to be the (unique) polynomial such that $\cos(n\theta) = T_n(\cos\theta)$. Show that $\{T_n(x)\}_{n=1}^{\infty}$ is an orthogonal set in the Euclidean space $(V, \langle \cdot, \cdot \rangle)$. What is $||T_n||$?

Q2 Let A, B be two $n \times n$ matrices. Prove that AB and BA have the same characteristic polynomial.

Q3 Let V be a 4-dimensional vector space, and let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ be a basis in V . Let A be the linear operator on V defined by

$$A\vec{v}_1 = \vec{v}_2, \ A\vec{v}_2 = \vec{v}_3, \ A\vec{v}_3 = \vec{v}_4, \ A\vec{v}_4 = \vec{v}_1.$$

(a) Find the matrix of A with respect to the basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$.

(b) Find the eigenvalues of A. How many eigenspaces does A have, and what are their dimensions?

(c) Find a basis of V consisting of eigenvectors of A.

Q4 Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & \alpha \end{bmatrix}.$$

(a) Show that A is diagonalizable **if and only if** $\alpha \neq 1$.

(b) For $\alpha \in (-1,1)$, find the eigenvalue decomposition of A, and compute A^k for $k \in \mathbb{N}$.

(c) For $\alpha \in (-1,1)$, show that $P = \lim_{k\to\infty} A^k$ exists and that P is a projector. What is the relationship between $R(P)$ and the eigenvectors of A?

Q5 Suppose that A, B, C are $n \times m$, $m \times p$, $m \times q$ matrices, respectively. If $AB = 0$ and $AC = 0$, and $rank(A) + rank(B) = n$, prove that there exists a $p \times q$ matrix D such that $C = BD$. In addition, prove that such D is unique if and only if $rank(B) = p$.