MASTER'S COMPREHENSIVE EXAM IN
Math 603-MATRIX ANALYSIS
January 2019

Do any three (out of the five) problems. Show all work. Each problem is worth ten points.

Q1 Let V be a complex finite dimensional inner-product space, and let $\mathbf{P} : V \to V$ be the projection on the subspace W_1 along the subspace W_2, where $V = W_1 \oplus W_2$.

(a) Show that $\mathbf{P}$ is an orthogonal projection (that is $W_2 = W_1^{\perp}$) if and only if $\|Px\| \leq \|x\|$ for all $x \in V$. (Hint: for $\Leftarrow$ part, let $w_1 \in W_1$ and $w_2 \in W_2$ be nonzero and $c \in \mathbb{R}$, and let $w = cw_1 + w_2$. Show that

$$2c\,\mathrm{Re}\,\langle w_1|w_2\rangle + \|w_2\|^2 \geq 0.$$

If $\langle w_1|w_2\rangle \neq 0$, derive a contradiction by a choice of c.)

(b) Show that if $\mathbf{P}$ is unitary, then it is in fact an identity matrix.

(c) Suppose that $\mathbf{P}$ is normal over $\mathbb{C}$. Prove that $\mathbf{P}$ is an orthogonal projection.

Q2 Let $\mathbf{D}$ be a diagonal matrix of size n with diagonal entries $d_1 < d_2 < \cdots < d_n$. Let Z be a symmetric rank 1 matrix with nonzero eigenvalue ρ and no zero entries. Prove that if λ is an eigenvalue of $\mathbf{D} + \mathbf{Z}$ and v is the corresponding eigenvector, then

(a) $\mathbf{D}$ and $\mathbf{D} + \mathbf{Z}$ do not have any common eigenvalues. (Hint: Assume λ is a common eigenvalue, write $\mathbf{Z}$ in terms of a vector z and show first that $\langle z|v\rangle = 0$.)

(b) $\mathbf{Z}v \neq 0$.

Q3 (a) For matrices $\mathbf{A}$ and $\mathbf{B}$ of the same shape show that $\mathbf{A}$ is row equivalent to $\mathbf{B}$ if and only if $R(\mathbf{A}^T) = R(\mathbf{B}^T)$.

(b) For matrices $\mathbf{A}_{m\times n}$ and $\mathbf{B}_{m\times p}$, prove that $R(\mathbf{A}\,|\,\mathbf{B}) = R(\mathbf{A}) + R(\mathbf{B})$.

(c) If $\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{pmatrix}$ is a square matrix such that $N(\mathbf{A}_1) = R(\mathbf{A}_2^T)$, prove that $\mathbf{A}$ must be nonsingular.

Q4 Let A be an $n \times n$ unitary complex matrix.

(a) Prove that the columns of a unitary matrix are orthonormal.

(b) Recall that an $n \times n$ complex matrix A is called lower triangular if its ijth entry $A_{ij} = 0$ for all $i < j$. Prove that if A is both unitary and lower triangular, then it must be a diagonal matrix. What is the absolute value/modulus of the iith entry A_{ii} of A?

Q5 Assume that the $n \times n$ real matrix A has characteristic polynomial

$$p(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$$

(a) Prove that A is invertible if and only if $a_0 \neq 0$.

(b) If A is invertible, prove that

$$span\{\ldots, A^{-2}, A^{-1}, I_n, A, A^2, \ldots\} = span\{I_n, A, A^2, \ldots A^{n-1}\}.$$

(c) Suppose that B is an $n \times n$ real matrix and ξ is an eigenvalue of $p(B)$. Prove that there exists an eigenvalue λ of B such that $p(\lambda) = \xi$.