MASTER'S COMPREHENSIVE EXAM IN
Math 600–REAL ANALYSIS
January 2023

Do any three (out of the five) problems. Show all work. Each problem is worth ten points. In the following, $\mathbb{R}^n$ carries the usual metric.

Q1 Let d be a metric on M. Define ρ on $M \times M$ by $\rho(x, y) := \min\{1, d(x, y)\}$. Assume that ρ is a metric on M. Show the following:

(a) A sequence (x_n) is Cauchy in (M, d) if and only if it Cauchy in (M, ρ).

(b) A sequence (x_n) is convergent in (M, d) if and only if it is convergent in (M, ρ).

(c) (M, d) is complete if and only if (M, ρ) is complete.

(d) Show that compact/connected sets are the same in both (M, d) and (M, ρ), i.e., a set is compact (resp. connected) in (M, d) if and only if it is compact (resp. connected) in (M, ρ). (*Hint*: consider the identity map.)

Q2 (a) Let (x_k) and (y_k) be two sequences in a compact set A in a normed vector space. For each k, let $z_k = \lambda_k x_k + (1 - \lambda_k) y_k$ for some $\lambda_k \in [0, 1]$. Show that (z_k) has a convergent subsequence.

(b) Let C be a closed, path-connected (=arcwise connected) set in (M, d) and $f : \mathbb{R} \to (M, d)$ be continuous. Suppose $C \cap f([-5, 1])$ is nonempty. Show that (i) $C \cap f([-5, 1])$ is compact; and (ii) $C \cup f([-5, 1])$ is path-connected.

(c) Let $I = [a, b]$ be an interval in $\mathbb{R}$ with $a < 0 < b$, and $f : I \to I$ be continuously differentiable on I, i.e., $f'(\cdot)$ is continuous on I. Show that (i) there exists $x_* \in I$ such that $L := \max_{x \in I} |f'(x)| = |f'(x_*)|$; (ii) $\frac{f(I)}{L+1} \subseteq I$; and (iii) the equation $(L + 1)x - f(x) = 0$ has a unique solution. (*Hint* for (iii): consider the function $\frac{f(x)}{L+1}$.)

Q3 Consider a differentiable function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ and $|f'(x)| \leq \Delta < \infty$ for all $x \in \mathbb{R}$. (For example, $f(x) = \sin x$ is such a function.) Let (t_n) be a sequence in $\mathbb{R}$ such that $\sum_{n=1}^{\infty} |t_n| < \infty$.

(a) Show that f is Lipschitz on $\mathbb{R}$.

(b) Show that the series $\sum_{n=1}^{\infty} f(t_n x)$ converges uniformly on any interval of the form $[a, b]$ in $\mathbb{R}$.

(c) For any $x \in \mathbb{R}$, let $F(x)$ denote the sum of the series $\sum_{n=1}^{\infty} f(t_n x)$. Show that F is differentiable and express the derivative as a series.

Q4 Consider the space $C[0, 1]$ of all real-valued continuous functions on the interval $[0, 1]$ equipped with the sup-norm $|| \cdot ||$. In $C[0, 1]$, consider

$$K = \Big\{ p : p(t) = a_0 + a_1 t + a_2 t^2,\ ||p|| \leq 1 \Big\}.$$

(So, K is the set of all quadratic polynomials with sup-norm less than or equal to one.)

(a) Show that there is a positive number Δ such that for any $p \in K$, $p(t) = a_0 + a_1 t + a_2 t^2$,

$$|a_0| + |a_1| + |a_2| \leq \Delta.$$

(*Hint*: Assuming the contrary, suppose $p^{(k)}$ is a sequence in K such that $|a_0^{(k)}| + |a_1^{(k)}| + |a_2^{(k)}| \to \infty$. Look at the sequence $q^{(k)} := \frac{p^{(k)}}{|a_0^{(k)}|+|a_1^{(k)}|+|a_2^{(k)}|}$.)

(b) Show that K is an equicontinuous family.

(c) Show that K is compact in $C[0,1]$.

Q5 Provide the definition of the (Fréchet) derivative of a map $f : V_1 \to V_2$ at a point $x \in V_1$, where $(V_i, \|.\|_i)$ are finite dimensional normed vector spaces.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$\begin{aligned} f(x,y) &= y^2 \sin(1/x) \qquad && \text{if } x \neq 0, \\ f(x,y) &= y^2 && \text{if } x = 0. \end{aligned}$$

(a) Decide if f has partial derivatives at $(0,0)$ and if so compute them.

(b) Decide if f has a Fréchet derivative at $(0,0)$ and if so what is it?

(c) What is the largest subset of $\mathbb{R}^2$ on which f is Fréchet differentiable?