MASTER'S COMPREHENSIVE EXAM IN
Math 603-MATRIX ANALYSIS
January 2023

Solve any three (out of the five) problems and indicate clearly which problems you wish to be considered. Show all work. Each problem is worth ten points.

Q1 (a) If $\mathbf{A}$ is a square matrix such that $\mathbf{I}-\mathbf{A}$ is nonsingular, prove that

$$\mathbf{A}(\mathbf{I}-\mathbf{A})^{-1} = (\mathbf{I}-\mathbf{A})^{-\mathbf{1}}\mathbf{A}.$$

(b) If $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{A}+\mathbf{B}$ are each nonsingular, prove that

$$\mathbf{A}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{B} = \mathbf{B}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{A} = (\mathbf{A}^{-1}+\mathbf{B}^{-1})^{-1}.$$

(c) Let $\mathbf{S}$ be a skew-symmetric matrix with real entries. If

$$\mathbf{A} = (\mathbf{I}+\mathbf{S})(\mathbf{I}-\mathbf{S})^{-1},$$

show that $\mathbf{A}^{-1} = \mathbf{A}^T$.

Q2 For the following problem <u>you may use the following result</u>:
If $\mathbf{A}, \mathbf{B}$ are $m \times n$ matrices, then

$$\text{rank}(\mathbf{A}+\mathbf{B}) \le \text{rank}(\mathbf{A}) + \text{rank}(\mathbf{B}).$$

(a) Prove that for any two $m \times n$ matrices $\mathbf{A}, \mathbf{B}$ we have

$$\text{rank}(\mathbf{A}+\mathbf{B}) \ge \text{rank}(\mathbf{A}) - \text{rank}(\mathbf{B}).$$

(b) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ be two non-zero vectors. Show that if $\mathbf{v}^T\mathbf{u} = -1$, then $\mathbf{A} = \mathbf{I} + \mathbf{u}\mathbf{v}^T$ is singular. Determine $\text{rank}(\mathbf{A})$.
Hint: Show that one of the two vectors lies in $N(\mathbf{A})$.

(c) With the notation from (b), show that if $\mathbf{v}^T\mathbf{u} \neq -1$, then $\mathbf{A}$ is non-singular and determine its inverse.
Hint: Search for an inverse of the form $\mathbf{B} = \mathbf{I} + \alpha\mathbf{u}\mathbf{v}^T$ with α a certain real number.

Q3 Assume $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{P}$ are square matrices.

(a) If $\mathbf{P}$ is nonsingular and $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$, explain why $\mathbf{B}^k = \mathbf{P}^{-1}\mathbf{A}^k\mathbf{P}$.

(b) If $\mathbf{A}$ and $\mathbf{B}$ are symmetric matrices that commute, prove that the product $\mathbf{AB}$ is also symmetric. If $\mathbf{AB} \neq \mathbf{BA}$, is $\mathbf{AB}$ necessarily symmetric? Prove or give a counterexample.

(c) For diagonalizable matrices, prove that $\mathbf{AB} = \mathbf{BA}$ if and only if $\mathbf{A}$ and $\mathbf{B}$ can be *simultaneously diagonalized*, i.e., there exists a nonsingular matrix $\mathbf{P}$ so that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}_1$ and $\mathbf{P}^{-1}\mathbf{B}\mathbf{P} = \mathbf{D}_2$.
Hint: If $\mathbf{A}$ and $\mathbf{B}$ commute, then so do $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} \lambda_1\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{pmatrix}$ and $\mathbf{P}^{-1}\mathbf{B}\mathbf{P} = \begin{pmatrix} \mathbf{W} & \mathbf{X} \\ \mathbf{Y} & \mathbf{Z} \end{pmatrix}$.

Q4 (a) Show that $\mathbf{A} \in \mathbb{R}^{n\times n}$ is normal and has real eigenvalues if and only if $\mathbf{A}$ is symmetric.

(b) For a normal matrix $\mathbf{A}$, explain why $(\lambda, \mathbf{x})$ is an eigenpair for $\mathbf{A}$ if and only if $(\overline{\lambda}, \mathbf{x})$ is an eigenpair for $\mathbf{A}^*$.

(c) If $\mathbf{A}$ is normal matrix with $\sigma(\mathbf{A}) = \{\lambda_1, \ldots, \lambda_k\}$, show that the eigenvectors corresponding to distinct eigenvalues are orthogonal, that is,

$$N(\mathbf{A} - \lambda_i \mathbf{I}) \perp N(\mathbf{A} - \lambda_j \mathbf{I}) \quad \text{for} \quad \lambda_i \neq \lambda_j.$$

Q5 Given two complementary subspaces X and Y of $\mathbb{R}^n$ ($X + Y = \mathbb{R}^n$, $X \cap Y = \{0\}$), a matrix $\mathbf{P} \in \mathbb{R}^{n\times n}$ is called a projector on X along Y if $R(\mathbf{P}) = X$ and $N(\mathbf{P}) = Y$. For the following problem <u>you may use the following results</u>:

- A matrix $\mathbf{P}$ is a projector if and only if it is idempotent, that is, $\mathbf{P}^2 = \mathbf{P}$; in this case the spaces are given by $X = R(\mathbf{P})$ and $Y = N(\mathbf{P})$.
- For all $u \in \mathbb{R}^n$, u can be written as $u = x + y$, with $x = Pu \in X$ and $y = (I - P)u \in Y$ being unique.

Given are two projectors $\mathbf{P}$ and $\mathbf{Q}$, both in $\mathbb{R}^{n\times n}$ with $X_1 = R(\mathbf{P})$ and $Y_1 = N(\mathbf{P})$ and $X_2 = R(\mathbf{Q})$ and $Y_2 = N(\mathbf{Q})$.

(a) Show that $\mathbf{P} - \mathbf{Q}$ is a projector if and only if $\mathbf{PQ} = \mathbf{QP} = \mathbf{Q}$.

(b) If $\mathbf{P} - \mathbf{Q}$ is a projector, then $R(\mathbf{P} - \mathbf{Q}) = X_1 \cap Y_2$ and $N(\mathbf{P} - \mathbf{Q}) = Y_1 + X_2$.