MATH 620 Numerical Analysis Comprehensive Exam – Jan 25, 2018

Attempt any 3 of the 4 questions below. Show your work.

1. (a) Let $g(x) = \frac{2}{3}(\cos(x) + \sin(x))$ and consider the iteration $x_{n+1} = g(x_n)$ for $n = 1, 2, \ldots$. Show that this converges to the same value α for any initial guess $x_0 \in \left[0, \frac{\pi}{2}\right]$.

   (b) Does the above iteration converge quadratically to α? If not, provide an alternative iterative formula for x_{n+1} in terms of x_n which will have quadratic convergence (briefly say why).

2. (a) Let $Q(f) = \sum_{i=1}^{5} w_i f(x_i)$ be the 5 point Gaussian quadrature rule for approximating $\int_{-1}^{1} f(x)\, dx$, i.e the one which has the maximum possible degree of precision. Let $P_i(x)$ be the Legendre polynomial of degree i and $l_i(x), i = 1, 2, \ldots 5$ be the Lagrange interpolating polynomials corresponding to the nodes $\{x_i\}$. Justifying your steps, calculate the values of each of the following (i) $Q(P_7)$ (ii) $Q(x^4 l_3)$ (iii) $Q(x^5 l_3)$. (You need to give actual numbers as answers.)

   (b) Which of the approximations in (a) give the exact value of the integral? Justify.

3. (a) Determine A, B, C so that the formula

   $$f'(x) \approx D[f](x) = Af(x) + Bf(x-h) + Cf(x+2h)$$

   has the highest possible order of accuracy.

   (b) Show that the error can be expressed in the form

   $$f'(x) - D[f](x) = Kf^{(k)}(\xi)h^p.$$

   Specify K, k, p and the interval in which ξ will lie.

4. (a) Consider the problem

   $$y' = 2x \quad y(0) = 0.$$

   Suppose y_n is the Euler's Method approximation at $x_n = nh$. Show that

   $$y_n = x_{n-1}x_n \quad n \geq 1.$$

   Hence show that the error for each fixed value of x_n is proportional to h.

   (b) Let $0 \leq a \leq \frac{1}{2}$ and consider the following generalized Trapezoidal Method:

   $$y_{n+1} = y_n + h\left[af(x_n, y_n) + (1-a)f(x_{n+1}, y_{n+1})\right] \quad n \geq 0.$$

   Show that this is absolutely stable.