Math 630 Comprehensive Examination

January, 2022

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.

1. (a) Compute the "full" singular value decomposition $A = U\Sigma V^T$ of the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & 2 \\ 4 & -2 & 4 \\ 0 & 0 & 0 \end{bmatrix},$$

Summarize your results clearly and check them!

(b) If you wish to find the singular value decomposition of the transpose A^T of A, can you guess the $A^T = \tilde{U}\tilde{\Sigma}\tilde{V}^T$ instead of computing it?

2. (a) Explain how would you solve a system $Ax = b$ if you knew the LU-factorization of A.

(b) Assume that L_k is that matrix of multipliers used to "introduce" zeros in column k of A. Using only matrix-algebra operations (addition, multiplication, inversion), write L_k using the identity matrix I, its column e_k, and the vector with multipliers $\ell_k = [0\ \dots\ 0\ \ell_{k+1,k}\ \dots \ell_{n,k}]^T$. (Vectors are regarded as matrices with one column.)

(c) Find (state explicitly) L_k^{-1} and show that $L_k L_k^{-1} = I$.

3. (a) Describe the Householder reflector operator, and show that for any vectors $x, y \in \mathbb{R}^n$ with $\|x\|_2 = \|y\|_2 \neq 0$, $x \neq y$, there exists a Householder reflector R so that $Rx = y$.

(b) Compute the QR factorization of the matrix

$$A = \begin{bmatrix} 2 & -2 \\ -3 & 0 \\ 6 & 1 \end{bmatrix}$$

using Householder reflectors. You may leave Q as a product of Householder transforms (by showing the vectors involved).

(c) Given $b = [1, -1, 1]^T \in \mathbb{R}^3$ and the matrix A from (b), solve the least squares (LS) problem

$$\min_{x \in \mathbb{R}^2} \|Ax - b\|_2^2 .$$

You may use either the method of normal equations or the QR factorization. Explain (briefly) why the LS problem has a unique solution.

4. (a) State the algorithm of the power method. Explain how you prevent overflow/underflow.

   (b) Let

$$A = \begin{bmatrix} 0.99 & 0 \\ 0 & 1 \end{bmatrix}.$$

   Find the eigenvalues of A and the associated eigenvectors.

   (c) Carry out power iteration starting with $q_0 = [1\ 1]^T$. Derive a general expression for q_j.

   (d) How many iterations are required in order to obtain $\|q_j - v_1\|_\infty / \|v_1\|_\infty < 10^{-6}$?