Math 630 Comprehensive Examination

January, 2023

**Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.**

1. Assume that a matrix $A$ has a nonsingular, unit $LU$ factorization ($L$ unit lower triangular and $U$ upper triangular and nonsingular). Also assume that the factorization is obtained using Type III operations, which allows to replace row $j$ by a linear combination of itself plus a multiple of another row $i$ with $i < j$.

   (a) Explain (briefly) how you would compute $A^{-1}$ efficiently by using the $LU$ factorization of $A$.

   (b) Show that for $k = 1, 2, \ldots$, the leading principal submatrix $A_k$ is nonsingular.

   (c) If the $(k+1)^{\text{th}}$ leading principal submatrix has the block structure

   $$A_{k+1} = \begin{pmatrix} A_k & b \\ c^T & \alpha_{k+1} \end{pmatrix},$$

   express the factors $L_{k+1}$ and $U_{k+1}$ in terms of the factors $L_k$ and $U_k$ (and their inverses) of $A_k$, the vectors $b$ and $c$, as well as the pivot $\alpha_{k+1} - c^T A_k^{-1} b$. Explain why the pivot is nonzero, and why all the pivots must be nonzero when $A$ has a nonsingular, unit $LU$ factorization.

   (d) If $A$ is a matrix that contains only integer entries and all of its pivots (diagonal entries in $U$) are 1, explain why $A^{-1}$ must also be an integer matrix.

2. Consider a nonsingular system $Ax = b$, for which there is some uncertainty in both $A$ and $b$. More precisely, the computed solution $\widetilde{x}$ satisfies $(A - E)\widetilde{x} = b - e$, where $\left\|A^{-1}E\right\| < 1$ for some matrix norm such that $\|I\| = 1$. Show that

   $$\frac{\|x - \widetilde{x}\|}{\|x\|} \leq \frac{\kappa}{1 - \kappa \frac{\|E\|}{\|A\|}} \left( \frac{\|e\|}{\|b\|} + \frac{\|E\|}{\|A\|} \right),$$

   where $\kappa = \|A\| \left\|A^{-1}\right\|$, and $\|v\|$ is a vector norm of a vector $v$ that is compatible with the matrix norm in the sense that $\|Av\| \leq \|A\| \cdot \|v\|$.
   Hint: If $B = A^{-1}E$, then $A - E = A(I - B)$ and $\alpha = \|B\| < 1 \Rightarrow \left\|B^k\right\| \leq \|B\|^k \to 0 \Rightarrow B^k \to 0$, so the Neumann series expansion yields $(I - B)^{-1} = \sum_{i=0}^{\infty} B^i$ so that

   $$\left\|(I - B)^{-1}\right\| \leq \frac{1}{1 - \|B\|},$$

   and utilize the identity $I - (I - B)^{-1} = -B\,(I - B)^{-1}$.

3. For (a) and (b) consider the Jacobi and Gauss-Seidel iterations for $2 \times 2$ systems of the form $Ax = b$ with

$$A = \begin{pmatrix} 1 & \alpha \\ \beta & 1 \end{pmatrix},$$

where $\alpha, \beta \in \mathbb{R}$.

   (a) Find a **necessary and sufficient condition** on the numbers $\alpha, \beta$ so that for any $b \in \mathbb{R}^2$, the Jacobi method converges to the solution of the system $Ax = b$.

   (b) Find a **necessary and sufficient condition** on the numbers $\alpha, \beta$ so that for any $b \in \mathbb{R}^2$, the Gauss-Seidel method converges to the solution of the system $Ax = b$.

   (c) Show that if an $n \times n$ matrix $A$ is nonsingular and upper triangular, then the Jacobi method converges in finitely many steps (assuming exact arithmetic).
   *Hint:* What kind of a matrix is the error propagator matrix?

4. Let

$$A = \begin{pmatrix} 8 & -18 \\ 3 & -7 \end{pmatrix} .$$

   (a) Compute the eigenvalues $\lambda_1, \lambda_2$ of $A$ (note that they are integers), with the notation convention that $|\lambda_1| \geq |\lambda_2|$. Starting with initial guess $q^{(0)} = [1, 1]^T$, compute two steps of the power method to find eigenvector approximations $q^{(1)}, q^{(2)}$. Use the Rayleigh quotient to compute an approximation $\lambda_1^{(2)}$ of the dominant eigenvalue $\lambda_1$ based on $q^{(2)}$. Estimate the convergence rate of $\lambda_1^{(n)}$ to $\lambda_1$ as $n \to \infty$.

   (b) Also starting with $p^{(0)} = [1, 1]^T$, compute two steps $p^{(1)}, p^{(2)}$ of the inverse iteration method with shift $\lambda = -2$. Use the Rayleigh quotient to compute the approximate eigenvalue $\mu_1^{(2)}$ based on $p^{(2)}$. If $p^{(n)}$ is the $n^{\text{th}}$ iterate of the shifted inverse iteration and $\mu_1^{(n)}$ is the corresponding Rayleigh quotient (still computed with $A$), what is

$$\mu_1 = \lim_{n \to \infty} \mu_1^{(n)} \text{ ?}$$

   Estimate the convergence rate of $|\mu_1 - \mu_1^{(n)}|$.