Masters Comprehensive Exam in Matrix Analysis (Math 603)
August 2013

Do any **three** problems. **Show all your work**. Each problem is worth 10 points.

1. Let V be a real inner-product product space and let V_1, V_2 be two subspaces of V. The addition of two vectors, the multiplication of a vector with a real number, and the inner product of two vectors in V are denoted by the usual symbols, e.g., $u+v$, $3u$, $\langle u, v\rangle$. The norm of a vector $v \in V$ is given by $\|v\| = \sqrt{\langle v, v\rangle}$.

a) Prove that the Cartesian product $W = V_1 \times V_2$ endowed with the operations

$$(u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2), \quad \forall\, u_1, v_1 \in V_1, u_2, v_2 \in V_2,$$
$$(\alpha + i\beta)(u_1, u_2) = (\alpha u_1 - \beta u_2, \beta u_1 + \alpha u_2), \quad \forall\, u_1, \in V_1, u_2 \in V_2, \alpha, \beta \in \mathbb{R},$$

is a complex vector space.

b) Is the functional $\langle\langle \cdot, \cdot \rangle\rangle : W \times W \to \mathbb{C}$ defined by

$$\langle\langle (u_1, u_2), (v_1, v_2) \rangle\rangle = \langle u_1, v_1\rangle + i\langle u_2, v_2\rangle$$

a complex inner-product on W? Explain why.

c) Show that the functional $\|\cdot\| : W \to \mathbb{R}_+$ given by

$$\|u\| = \|(u_1, u_2)\| = \sqrt{\langle u_1, u_1\rangle + \langle u_2, u_2\rangle}, \quad \forall\, u = (u_1, u_2) \in W,$$

is a norm on W.

d) Prove that there is an inner-product $\langle \cdot, \cdot \rangle$ on W which induces the norm defined in c), i.e., $\|v\|^2 = \langle v, v\rangle, \forall\, v \in W$.

2. Consider the vectors

$$a = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \; u = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \; v = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \; w = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \; e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

a) What is the dimension of the subspace $H = Span\{u, v, w\}$?
b) Find the orthogonal projection of a on H.
c) Find $\min_{x \in H} \|a - x\|_2$.
d) Let A be a 3×4-matrix such that $Au = 0,\ Aw = 0,\ Ae_1 = [1, -2, 3]^T,\ Ae_2 = [-2, 4, -6]^T$.
What are the nullspace and the column space of A^T?

3. Let u and v be two (column) vectors in $\mathbb{R}^n$ with all components positive and let $\langle u, v\rangle$ denote the usual inner product.

(a) Find all eigenvalues of the matrix uv^T.

(b) Find all eigenvalues of $I_n + uv^T$, where I_n is the $n \times n$ identity matrix.

(c) Show that all off-diagonal entries of $(I_n + uv^T)^{-1}$ are negative.

4.

(a) Let A be an $n \times m$ matrix and B be an $m \times p$ matrix. Suppose $rank(AB) = m$. Prove that $rank(A) = rank(B) = m$.

(b) Let A be an $n \times n$ matrix. Prove that $A^2 = I_n$ if and only if $rank(I_n + A) + rank(I_n - A) = n$.

5. Let A be an $n \times n$ real-valued matrix and $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be a basis of $\mathbb{R}^n$. Suppose that α_i is an eigenvector of A for each $i = 1, \cdots, n$ and A has different eigenvalues. Prove that there exists a vector v in $\mathbb{R}^n$ such that $\{v, Av, A^2v, \cdots, A^{n-1}v\}$ is also a basis of $\mathbb{R}^n$.