

2 + 2 = 2 × 2

THAT LITTLE EQUATION is one of the things I found fascinating about math when I was young. You might have felt the same way. When we were a bit older (and “math” became “mathematics”), we realized that there are only two solutions to $a + a = a \times a$. They are $a = 0$, and $a = 2$. It was a sign of our mathematical growth. Eventually we saw that the number of solutions to a problem, 0, 1, 2, ..., ∞ could be as interesting as the solutions themselves—sometimes more so.

Another area of mathematical growth is to try to generalize: We see something interesting and we try to fit it into a bigger context. Doing so helps us remember it, understand it, and, if we are very lucky, discover something new and interesting. But generalizing from one equation is like drawing a line given just one point; we don't know which of many different directions to go in.

But then we notice the equation:

$$1 + 2 + 3 = 1 \times 2 \times 3$$

Suddenly a direction for generalization starts to emerge from the fog. We can look for solutions to equations like $a + b + c + \dots + z = a \times b \times c \times \dots \times z$. That is, are there numbers whose sum is equal to their product? So we see it was not the equations we were trying to generalize, but rather the question.

Initially the question was implicit, or maybe it was vague. It is the second example that gave it some context. In some cases it may take more than two cases to nail down an interesting generalization. Here two seems to work. But our question still needs to be refined.

- Do we allow repeats? YES, of course we do; our original example involves two 2's.

- Do we allow 0? NO, 0's lead to trivial solutions.
- Do we allow fractions? NO, fractions would give infinitely many solutions to $a + b = a \times b$.
- Do we allow negatives? NO, negatives allow for solutions that are negatives of one another, for example -1, -2, and -3 would be a solution to $a + b + c = a \times b \times c$.
- Finally, rearrangements of a solution will be counted as the same solution. Equivalently, we will only accept solutions with $a \leq b \leq c \leq \dots \leq z$.

We have come quite far. Let's see how well we can solve our generalized problem. First we restate it as:

For $n \geq 2$, find positive integers $a_1 \leq a_2 \leq \dots \leq a_n$ such that:

$$\sum_{i=1}^n a_i = \prod_{i=1}^n a_i$$

Here's a warm-up problem:

1. How many solutions are there for $n = 2$ and for $n = 3$?
The main problems:
2. Show that there is at least one solution for all n . If you want a hint, see below just above the solution to last issue's puzzle.
3. Extend problem 1: Find the number of solutions for each n for $n = 4$ to 15.

Going Further

This brings me to a bit of a confession. We really didn't have to throw out fractional solutions. But if we had left them in, it would have led us to a different generalization and a different problem to solve. Indeed, what really started me thinking about the problems posed above was a problem I saw in a recent issue of

The American Mathematical Monthly, Vol. 122, No. 3, March 2015, page 284. Problem 11,827 reads:

Show that there are infinitely many rational triples (a, b, c) such that $a + b + c = abc = 6$.

In this problem $n = 3$ and the common value of the sum and product is fixed at 6; in our posed problem we let these vary. On the other hand, while in our posed problems we restricted solutions to positive integers, in the *Monthly* problem all rational numbers can be used in the solutions. And as a result of these tradeoffs, the *Monthly* problem asks for infinitely many solutions. So far I have been able to find only two: (1, 2, 3) and (-1.5, -0.5, 8). Can you discover others? I'd like to see them. (We will learn the answer to this problem in about two years given the *Monthly's* publishing schedule. You will find out the answer to our posed problems in about two months, if you haven't already figured them out.)

So we have two interesting generalizations of the equation in the title. One generalization underlies the posed problems, the other underlies the problem in the *Monthly*. My guess is that there are more. It would be interesting to read about any other mathematical generalizations you may think of. So please send them in; if I like yours, I may use it (and give you credit, of course).

[Hint for Problem 2: Try to generalize the solutions for $n = 2$ and $n = 3$ to all n .]

Solution to Previous Puzzle

1. What is the probability that my youngest brother sees a family win a car in the first episode of *Family Feud* that he watches?

One could solve this as a Markov Chain, or you can realize that the probability of seeing a family win its fifth match in a row is one-half the probability of a family winning its fourth match, which is one-half of winning its third match, etc. If families didn't have to

Editor's note: This is the inaugural puzzle by new puzzle co-editor Stephen Meskin. Solutions may be emailed to cont.puzzles@gmail.com. In order to make the solver list, your solutions must be received by Sept. 30, 2015.

retire after their fifth win in a row, the probability of seeing a fifth win would be $(1/2)^5 = 1/32$, and the probability of seeing a family win six or more in a row is also $1/32$. But since a family can't win more than five in a row, we need to subtract $1/32$ from the denominator, leaving us with a probability of $(1/32)$ divided by $(31/32)$, which equals $1/31$.

2. What is the probability my middle brother sees two families win a car in 10 consecutive episodes?

There are 15 episode pairs where it's possible my brother sees two families winning a car: episodes 1 and 6, 1 and 7, 1 and 8 ... all the way to episodes 6 and 10. We need to sum the probabilities of these pairs occurring to find our answer. For five of these pairs, including episodes 1 and 6, the probability is $(1/31) \times (1/2)^4$. (Notice that either family can win episode 2, so that's why the exponent is 4 and not 5.) The $(1/31)$ part of the equation is the probability of him seeing a car won on episode 1, our answer to the first puzzle.

In a vacuum, the probability of winning a car is the same in every episode, which means that the probability of him seeing a family win a car on episode 2, 3, 4, or 5 is also $1/31$. For the remaining 10 of these pairs, including episode 1 and 7, the probability is $(1/31) \times (1/2)^5$. In this case, it doesn't matter who wins episode 2, but a family must win for the first time in episode 3, and then win out the remaining four episodes. Hence the exponent is 5. The sum of the probabilities is therefore: $10 \times (1/31) \times (1/2)^5 + 5 \times (1/31) \times (1/2)^4 = 2.0\%$.

Solvers

Christopher Allard, Robert Bartholomew, Bob Byrne, William Carroll, Bob Conger, Andrew Dean, Bernie Erickson, Mark Evans, Bill Feldman, Yan Fridman, Renauld Guilbert, Rui Guo, Eric Kovac, Chi Kwok, David Lovit, Harold Luber, Lee Michelson, Paul Navratil, David Promislow, Craig Schmid, Matt Sedlock, Lenny Shteyman, John Snyder, Al Spooner, and Daniel Wade. Robert Bartholomew was accidentally left off the solvers list for the previous puzzle.

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