Master's Comprehensive Exam in Math 603

January 2014

Do any three (out of the five) problems. Show all work. Each problem is worth ten points.

1. Let A be an $m \times n$ matrix and B be an $m \times p$ matrix. Let $C = \big[A \,|\, B\big]$ be an $m \times (n+p)$ matrix.

   (a) Show that $R(C) = R(A) + R(B)$, where $R(\cdot)$ denotes the range of a matrix.

   (b) Show that $\text{rank}(C) = \text{rank}(A) + \text{rank}(B) - \dim(R(A) \cap R(B))$.

2. Let X and Y be two *distinct* p-dimensional subspaces of an n-dimensional inner product space V with $0 < p < n$. Suppose that for any nonzero vectors $x \in X$ and $y \in Y$, the inner product $\langle x, y \rangle \neq 0$.

   (a) Show that $\dim(X \cap Y) = 0$.

   (b) Let $X^\perp$ be the orthogonal complement of X. Show that $V = X^\perp \oplus Y$.

3. Let X and Y be complementary subspaces of $\mathbb{R}^n$, i.e., $\mathbb{R}^n = X \oplus Y$. Let the $n \times n$ matrix P denote the projector onto X along Y. Let B_X and B_Y be respective bases for X and Y.

   (a) Express P in terms of B_X and B_Y, and show that the projection matrix P is independent of bases B_X and B_Y.

   (b) Suppose $\det(P) = 1$. Determine the subspace Y.

   (c) Show that $\text{rank}(P) = \text{trace}(P) = \dim(X)$.

   (d) Show that $\|P\|_2 \geq 1$ if $P \neq 0$, where $\|\cdot\|_2$ denotes the induced matrix 2-norm. When is $\|P\|_2 = 1$?

4. Let A be an $n \times n$ matrix. If there exists $k > n$ such that $A^k = 0$, then

   (a) prove that $I_n - A$ is nonsingular, where I_n is the $n \times n$ identity matrix;

   (b) show that there exists $r \leq n$ such that $A^r = 0$.

5. Solve the following problems.

   (a) Let $T_i \in \mathbb{R}^{n \times n}$ be upper triangular matrices with $[T_i]_{ii} = 0$ for each $i = 1, \ldots, n$. Determine $T_1 \cdot T_2 \cdots T_n$.

   (b) Let B be an $m \times n$ matrix. Show that $A := I + B^T B$ has eigenvalue one (i.e., $1 \in \sigma(A)$) if and only if the columns of B are linearly dependent.

   (c) Show that a symmetric matrix A is positive semidefinite if and only if there exists a square matrix P such that $x^T A x = \sum_{i=1}^n z_i^2(x)$, where $z(x) := (z_1(x), \ldots, z_n(x))^T = Px$.