COMPREHENSIVE EXAMINATION

Math 650 – Optimization
August 1999
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INSTRUCTIONS:
You must do either Question 1 (25 points); Question 2 or 3 (20 pts.); Question 4 (20 pts.); Question 5 (15 pts.); and Question 6 or 7 (20 pts.). The exam is worth 100 points. **You must show all your work for full credit!**

1. Consider the optimization problem

$$\begin{array}{ll} \min & \ln x - y \\ \text{subject to} & x^2 + y^2 \leq 4 \\ & x \geq 1. \end{array}$$

(a) Sketch the constraint set.

(b) Write down the Fritz John (FJ) conditions. Show that all points satisfying the FJ conditions must also satisfy the Karush–Kuhn–Tucker (KKT) conditions.

(c) Write down the KKT conditions. Then determine all points satisfying these conditions. *Hint: consider all the different possibilities with λ_i positive or zero.*

(d) Determine whether each KKT point satisfies the second order necessary/sufficient conditions. Consequently use these results to determine the local and global optimizers of the problem.

2. Let Q an $n \times n$ symmetric, positive definite matrix, $a \in \mathbb{R}^n$, and $a \neq 0$, $c \in \mathbb{R}$ be given. Consider the optimization problem

$$\begin{array}{ll} \min & \frac{1}{2}\langle x, Qx\rangle \\ \text{subject to} & \langle a, x\rangle \leq c. \qquad (P) \end{array}$$

(a) Show that (P) is a convex programming problem.

(b) Show that if $c \geq 0$, then $x^* = 0$ is the unique optimal solution to (P).

(c) Now suppose $c < 0$. Determine the dual problem to (P).

(d) Use (c) to determine the optimal solutions to both (D) and (P).

3. Let A an $m \times n$ matrix and $p \in \mathbb{R}^n$. Consider the linear programming problem

$$\begin{array}{ll} \min & t \\ \text{subject to} & Ay = 0, \qquad (P) \\ & p^T y - t = -1, \\ & y \geq 0, t \geq 0. \end{array}$$

(a) Formulate the dual program (D) to (P) as an explicitly written linear program.

(b) Show that both (P) and (D) have optimal solutions. *Hint: use LP duality.*

(c) Let v^* be the common optimal objective value of (P) and (D). Show that it follows from (P) and (a) that $0 \leq v^* \leq 1$. In fact, show that v^* is equal to either zero or one. *Hint: use complementarity for the last part.*

4. (a) Let $C \subseteq \mathbb{R}^n$ be a non–empty convex set such that $C \cap \mathbb{R}^n_+ \subseteq \{0\}$, where $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geq 0\}$ is the non–negative orthant. Prove that there exists $p \in \mathbb{R}^n$, $p \geq 0$, $p \neq 0$ such that $\sup_{x \in C} \langle p, x \rangle \leq 0$.

(b) Use an argument similar to the one in (a) to show that there exists $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, not both zero such that

$$\lambda_1 x_1 + \lambda_2 x_2 \geq 0$$

for all (x_1, x_2) satisfying $x_1^2 + x_2^2 \leq 2$, $(x_1 - 3)^2 + x_2^2 \leq 8$, and $x_2 \leq \sqrt{7}/2$.

5. Consider the "diamond" D in the plane with vertices at the points $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$. Describe D as the intersection of four inequalities. Use this to determine the polar D of, D^* where

$$D^* = \{y \in \mathbb{R}^2 : \langle x, y \rangle \leq 1, \forall x \in D\}.$$

Hint: Farkas Lemma.

6. Let $f : C \to \mathbb{R}$ be a convex function where $C \subseteq \mathbb{R}^n$ is an open, convex set. Prove that f is a continuous function on C.

7. Consider the optimization problem (P)

$$\min\{f(x) : g_i(x) \leq 0, i = 1, \ldots, m\}$$

where f, g_i are continuously differentiable function define on $\mathbb{R}^n$.

(a) Write down the Fritz John (FJ) conditions for a local minimizer x^* of (P).

(b) Suppose that the Mangasarian–Fromovitz (MF) conditions:

$$\exists d \in \mathbb{R}^n \text{ such that } \langle \nabla g_i(x^*), d \rangle < 0, \quad \forall i \in I(x^*),$$

where $I(x^*)$ is the set of active constraints at x^*. Show that the KKT conditions are satisfied at x^*.

(c) Suppose that g_i are convex and that (P) satisfies the *Slater condition*, that is, there exists a point x_0 such that $g_i(x_0) < 0$, $i = 1, \ldots, m$. Show that the KKT conditions are satisfied at any FJ point x^*. *Hint: show that MF conditions hold true.*