COMPREHENSIVE EXAMINATION
Math 650 – Optimization
January 2000
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INSTRUCTIONS:
You must answer Questions 1 (30 points), 2 (35 points), and 3 (15 points). Choose either Question 4 or 5 (20 points each). The exam is worth 100 points. **You must show all your work for full credit!**

Q1. Consider the convex programming problem

$$\begin{array}{rll} \min & (x_1-3)^2+(x_2-2)^2 & \\ \text{subject to} & x_1^2+x_2^2 \le 5 & \\ & x_1+2x_2 \le 4 & (P) \\ & x_1 \ge 0, x_2 \ge 0. & \end{array}$$

(a) Note that the optimal solution to the above problem is the projection from a certain point in the plane to the convex region defined by the constraints. Sketch the constraint region and use it to numerically determine the optimal solution $x^* = (x_1^*, x_2^*)$.

(b) (i) Write down the Lagrangian function for the optimization problem; (ii) write down the Karush–Kuhn–Tucker (KKT) conditions; (iii) use the KKT conditions to numerically determine the Kuhn–Tucker (Lagrange) multipliers $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ at the optimal solution x^* in (a);

(c) (i) Use the Lagrangian in (b) to determine the dual program (D) explicitly as a maximization problem in which only the variables λ appear.

(d) Give a theoretical reason as to why the duality gap $\min(P) - \sup(D)$ is zero, or if it is not, explain why not. (You may use a theorem proved in class relevant to the situation at hand, but state carefully the theorem.)

Q2. Consider the optimization problem (P) $\min\{f(x) : g_i(x) \le 0, i = 1, \ldots, m\}$ in which all the functions f, g_i are differentiable. Denote the feasible region $\mathcal{F} = \{x : g_i(x) \le 0, i = 1, \ldots, m\}$.

(a) Write down the Fritz John (FJ) conditions and the KKT conditions at a local minimizer x^* of (P).

(b) Suppose that the functions $g_i(x)$ are convex, and that there exists a feasible point $x^0 \in \mathcal{F}$ such that $g_i(x^0) < 0$, $i = 1, \ldots, m$. Prove that a constraint qualification is satisfied, so that the KKT conditions are satisfied at any local minimizer $x^* \in \mathcal{F}$.

(c) Now, consider the optimization problem in Problem 1 above, except that the objective function is now $\min -(x_1-3)^2 - (x_2-2)^2$ (that is, we are now maximizing the objective function in Problem 1). Use the results of (b) to show that the KKT conditions are satisfied at each local optimizer.

(d) Show that the KKT conditions are satisfied at the point (i) $(0,0)$; (ii) $(0,2)$.

(e) State second order *necessary* and *sufficient* conditions for optimality at a KKT point. Use these to investigate whether (i) the 2nd order sufficient conditions are satisfied at (0,0); (ii) whether the 2nd order necessary and sufficient conditions are satisfied KKT conditions at (0,2).

Q3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function with continuous second order partial derivatives. Consider the following three conditions: (i) f is convex; (ii) $f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle$ for all $x, y \in \mathbb{R}^n$; (iii) the Hessian $Hf(x)$ is positive semidefinite for all $x \in \mathbb{R}^n$. Prove the following results.

(a) (i) implies (ii)

(b) (ii) implies (i)

(c) (iii) implies (ii).

Q4. Recall that if K is a convex body (that is, a compact convex set) in $\mathbb{R}^n$ containing the origin in its interior, then its *polar* is defined to be the convex body

$$K^* := \{c \in \mathbb{R}^n : \langle c, x \rangle \leq 1 \quad \forall x \in K\}.$$

Now consider the triangle

$$K = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 \leq 1, -x_1 + x_2 \leq 1, -x_2 \leq 1\}.$$

Prove that its polar is given by

$$K^* = \{y_1(1,1) + y_2(-1,1) + y_3(0,-1) : y_1 + y_2 + y_3 \leq 1, y_i \geq 0, i = 1,2,3\}.$$

Consequently, show $K^* = conv(\{(1,1), (-1,1), (0,-1)\})$ is the triangle with the vertices $(1,1)$, $(-1,1)$, and $(0,-1)$.

Hint: Interpret K^* as one part of Farkas Lemma: if (x_1, x_2) satisfies the three linear inequalities in the definition of K, then (x_1, x_2) must also satisfy $c_1 x_1 + c_2 x_2 \leq 1$. Alternatively, use linear programming duality.

Q5. Consider the set $K := \{x : Ax < 0\}$ where A is an $m \times n$ matrix.

(a) Show, by elementary arguments (that is, using no convexity), that $K = \emptyset$ if and only if

$$\{y \in \mathbb{R}^m : y_i < 0, i = 1, \ldots, m\} \cap \{Ax : x \in \mathbb{R}^n\} = \emptyset. \qquad (1)$$

(b) Assuming (1) is true, show by a separation argument, that there exists $a \in \mathbb{R}^m$ satisfying

$$a \geq 0, \quad a \neq 0, \quad A^T a = 0. \qquad (2)$$

(c) Combine (a) and (b) to prove that $\{x : Ax < 0\} = \emptyset$ if and only if the zero vector is in the convex hull of the rows of A.

Remark: (c) is Gordan's lemma used in proving Fritz John's conditions.