COMPREHENSIVE EXAMINATION

Math 650 / Optimization / August 2002

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Name ______________________________ SSN ______________________________

INSTRUCTIONS:
Solve any *two* out of the following three problems. You must *mark* clearly which two problems you would like to be graded. Each problem carries equal weight.

1. Consider the optimization problem

$$\begin{array}{rl} \min & -(x-4)^2 - y^2 \\ \text{subject to} & y^2 \le x, \\ & xy \le 8. \end{array}$$

(a) Sketch the feasible region. Show that the problem has no global optimal solution.

(b) Write down the Fritz John (FJ) conditions, and give a careful argument to prove that any FJ point must also be a KKT point.

(c) Determine all the KKT points. A useful fact: there is no KKT point in which the only non–zero Lagrange multiplier is the one corresponding to the last constraint.

(d) Test the second order necessary and sufficient conditions at *one* KKT point found in (c). What conclusions can you draw from these tests about the local optimality of this KKT point?

2. Consider the quadratic program

$$\begin{array}{ll} \min & \frac{1}{2}\langle Hx, x\rangle + \langle c, x\rangle \\ \text{s.t.} & Ax = b, x \geq 0, \quad (P) \end{array}$$

where A is an $m \times n$ matrix, and H is an $n \times n$ is a positive semi–definite matrix.

(a) Write the Fritz John optimality conditions for (P). Does the Karush–Kuhn–Tucker conditions hold true for this problem? Give a theoretical justification.

(b) Are the above conditions in (a) sufficient for optimality? Explain.

(c) Assume that $-\infty < \inf(P) < +\infty$. State the strong duality for the pair (P) and its Lagrangian dual (D), and explain whether it holds true. Give a complete justification for your answer.

Assume from now on that H is positive definite.

(d) Show that (P) has an optimal solution.

(e) Formulate the dual problem explicitly, that is, write its objective function and constraints in terms of only the dual variables.

(f) Assume that you somehow determined the optimal solution to (D). Using this knowledge, give a formula for the optimal solution to (P).

3. In the program

$$\min\{f(x) : x \in C\}, \quad (P),$$

C is a convex set in $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function.

(a) Show that any local minimizer of (P) is a global minimizer of f over C.

(b) Suppose that f is a differentiable. State and prove the *variational inequality* for a minimizer x^* of (P).

(c) Now, suppose that C is a polyhedron,

$$C = \{x : \langle a_i, x\rangle \le \beta_i, i = 1, \ldots, m, \langle b_j, x\rangle \le \gamma_j, j = 1, \ldots, k\}.$$

Show that Farkas Lemma applied to the variational inequality in (b) proves that any FJ point of (P) is a KKT point.