COMPREHENSIVE EXAMINATION
Math 650 / Optimization / August 2004
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Name ______________________________

INSTRUCTIONS:
You *must* solve Problem 1. You must also solve *one* problem from the set $\{2, 3\}$, but please *mark* clearly which of these two problems you would like to be graded. Each problem carries equal weight.

1. Consider the following constrained optimization problem

$$\begin{array}{ll} \min & x^2 + (y-3)^2 \\ s.t. & 2x^2 - y \le 0 \\ & -x^2 + y \le 1 \end{array}$$

Sketching the feasible region may a good idea to visualize the problem.

a. Write (display) the Fritz John conditions for the critical point(s) of the problem.

b. Give a geometric or analytical proof that every Fritz John point is a KKT point (that is, $\lambda_0 > 0$), and *display* the KKT conditions.

c. Show whether the points $(\mp 1, 2)$ are KKT points. Use second order conditions (necessary and/or sufficient) to determine the optimality status of these two points.

d. Repeat part **c.** for the point $(0, 1)$.

e. *(Extra Credit, 5 points)* Determine whether there are additional KKT points satisfying the constraint $y = x^2 + 1$. If there are, find them, and repeat part **c.** to determine their status.

2. This problem is about the structure of **convex** polyhedral objects.

a. Give concise definitions for a *polyhedral cone, a polytope, and a polyhedron.*

b. Show that the Minkowski sum of two polytopes is a polytope. That is, if P_1 and P_2 are polytopes, then the sum

$$P = P_1 + P_2 = \{x_1 + x_2 : x_i \in P_i, i = 1, 2\}$$

is a polytope.

c. Let K_1, K_2 be convex polyhedral cones. Show that their Minkowski sum $K = K_1 + K_2$ is also a convex polyhedral cone.

(d) A linear image of a polyhedron is a polyhedron, that is, if $C \subset \mathbb{R}^n$ is a polyhedron and $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, then $T(C)$ is a polyhedron.

Hint: There are different ways to express each polyhedral object.

3. Consider the linear program

$$\begin{array}{rl} \min & 4x + 3y - 4z \\ s.t. & x - 2y + 3z \leq 15 \\ & 2x - y + z \leq 2 \\ & x \geq 0, y \geq 0, z \geq 0 \end{array} \qquad (P)$$

Note that $(x, y, z) = (0, 0, 2)$ is feasible for (P).

The object of this problem is to compute the dual of (P) using *Lagrangian duality theory.* If you the problem in any other way (for example, using Farkas Lemma), you will receive *no* credit. However, you can compare your answers using different methods.

a. Write the Lagrangian function $L(x, y, z, \lambda)$ for (P). Use L to express (P) and its dual (D) as minimax (or maximin) problems.

b. Write (D) as a linear program which does not involve x, y, z. Can you determine whether (D) is feasible or not?

c. State the *weak duality theorem* for (P) and (D). Is this theorem true for the pair (P)–(D)? Explain in full; be succinct.

d. State the *strong duality theorem* for (P)–(D). Is this theorem true here? Explain in full; be succinct.