Math 620 Qualifying Exam 8/24/2011

Name:

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.

1. Using the unevenly spaced points $x_0 < x_1 < x_2$ where $x_1 - x_0 = h$ and $x_2 - x_1 = \alpha h$, establish the formula

$$f''(x) \approx \frac{2}{h^2}\left[\frac{f(x_0)}{(1+\alpha)} - \frac{f(x_1)}{\alpha} + \frac{f(x_2)}{\alpha(\alpha+1)}\right] \tag{1}$$

in the following two ways:

   (a) Approximate $f(x)$ by the Newton form of the interpolating polynomial of degree 2.

   (b) Calculate the undetermined coefficients A, B, and C in the expression $f''(x) \approx Af(x_0)+Bf(x_1)+Cf(x_2)$ by making it exact for the three polynomials 1, $x - x_1$, and $(x - x_1)^2$.

2. Consider the following data I_n obtained by some quadrature rule that approximates the true value denoted by I:

n	I_n
2	0.28451779686
4	0.28559254578
8	0.28570248748
16	0.28571317731
32	0.28571418363
64	0.28571427643

   (a) Based on the given data, predict the order of convergence p, that is, assuming that the error satisfies $I - I_n = c/n^p$ for some constant c independent of n, then what is a good estimate for p you can obtain based on the given data?

(b) Using your value of p from part (a), predict the value of c. [Hint: Start with $I_n - I_{n/2} = (I - I_{n/2}) - (I - I_n) = \cdots$ and notice that all quantities except c are known.]

(c) Using your predicted values of c and p, estimate how large n needs to be to guarantee that the error is less than the tolerance 10^{-11}?

Note: For each part, you need to derive appropriate formulas first. These carry the greatest weight for the scoring of this problem, greater than the calculations with numerical values.

3. Derive the *local truncation error* $\tau_n(Y)$ of the numerical method

$$y_{n+1} = y_n + \frac{h}{24}\left[9f(x_{n+1}, y_{n+1}) + 19f(x_n, y_n) - 5f(x_{n-1}, y_{n-1}) + f(x_{n-2}, y_{n-2})\right], \quad n \geq 2,$$

for the ordinary differential equation $y'(x) = f(x, y(x))$. Be sure to carry enough terms in the Taylor expansions to get the precise coefficient of the leading order of h. What is the order of consistency of the method? [Be sure to distinguish carefully between the *truncation error* $T_n(Y)$ and the *local truncation error* $\tau_n(Y) = T_n(Y)/h$, which defines the order of consistency.]

4. Consider the nonlinear system

$$\begin{cases} x = \frac{1}{2} + \frac{\sin(x+y^2)}{8}, \\ y = 2 + \frac{\ln(1+x^3+y)}{5}. \end{cases} \tag{2}$$

(i) Use the contraction mapping theorem to show that (2) has a unique solution $(x^*, y^*) \in D = [0,1] \times [0,3]$.

(ii) Use initial guess $(x_0, y_0) = (0, 0)$ to compute the iterates (x_1, y_1) and (x_2, y_2) obtained by applying the fixed point method to (2).

(iii) Estimate the number n of iterations needed for the n^{th} fixed point method iterate to satisfy

$$\|(x_n, y_n) - (x^*, y^*)\|_\infty \leq 10^{-8}.$$