Math 630 Qualifying Exam 8/24/2011

Name:

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.

1. (a) Show that for any induced matrix norm, $\kappa(A) \geq 1$.

   (b) If $Ax = b$ and $(A + \delta A)(x + \delta x) = b$, prove the inequality

   $$\frac{||\delta x||}{||x + \delta x||} \leq \kappa(A)\frac{||\delta A||}{||A||}$$

   where $\kappa(A)$ is the condition number of the matrix A.

   (c) Is the inequality in Part b above what we need to estimate the impact of perturbations in the matrix on perturbations in the solution? Why or why not?

   (d) Verify the inequality for the system

   $$\begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

   using

   $$\Delta A = \begin{pmatrix} 0 & 0 & 0.00003 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

   (e) Is the determinant of a matrix a good measure of the condition of a matrix? Give an example to help justify your answer.

2. Solving $Ax = \lambda x$ is equivalent for certain matrices to finding stationary points of the Rayleigh quotient.

(a) For which type of matrices is the above statement true?

(b) Write out the Rayleigh quotient R for the matrix

$$A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}.$$

(c) For the matrix A above, calculate $\partial R/\partial x_1$ and $\partial R/\partial x_2$.

(d) Which vectors x minimize and maximize this Rayleigh quotient?

(e) What subspace results from this gradient direction?

(f) The conjugate gradient method is also an optimal way to minimize and maximize R. What trick does the conjugate gradient algorithm use to achieve a better rate of convergence than steepest descent? Explain your answer in both algebraic and geometric terms. (Hint: explain why the conjugate gradient method is able to converge to a solution of the linear system $Ax = b$ in n iterations if $A \in \mathbb{R}^{n \times n}$.)

3. Prove that the vector norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent in $\mathbb{R}^n$, that is, show that

$$c_1 \|x\|_2 \le \|x\|_1 \le c_2 \|x\|_2 \quad \text{for all } x \in \mathbb{R}^n.$$

with $c_1 = 1$ and $c_2 = \sqrt{n}$. Demonstrate that the inequalities are sharp, that is, specify example vectors $x^{(\ell)}$ and $x^{(r)}$ such that the left and right inequality, respectively, becomes an equality.

4. Let $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$ be non-zero, and define the rank-one matrix $A = uv^T \in M_{m \times n}$.

(i) Compute a singular value decomposition of A, and determine $\|A\|_2$. (Hint: define $u_1 = u/\|u\|_2, v_1 = v/\|v\|_2$ and complete each of u_1, v_1 to an orthonormal basis in their spaces.)

(ii) Compute the pseudoinverse A^+ of A.

(iii) Briefly explain the relevance of A^+ to the question of finding the least-norm solution of the least-squares problem

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_2 ,$$

for a general matrix $A \in M_{m \times n}$ and $b \in \mathbb{R}^m$. Given $b = [1, \dots, 1]^T \in \mathbb{R}^m$ and A as in (a), find (in terms of the vectors u, v) the least-norm solution of the least-squares problem

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_2 .$$