Math 630 Qualifying Exam 1/19/2012

Name:

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.

(1a) If

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

what are the singular values of A?

(b) If the matrix of left singular vectors is

$$V = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

what is the minimum length least squares solution x^+ to $Ax = b$ if

$$b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}?$$

(c) Prove the theoretical formula you used in Part (b) gives the minimum norm least squares solution.

(2a) If you apply the power method to the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix},$$

what will be the rate of convergence? Is this a good or bad rate of convergence?

(b) Assuming you do not know the eigenvalues of A, explain very clearly how you would try to speed up the convergence of the power method to the eigenvector corresponding to the smallest eigenvalue. (Give as much detail as possible about your new algorithm.)

(c) Prove that in the shifted QR algorithm, A_{k+1} is unitarily similar to A_k.

(3) (a) Briefly describe the Cholesky factorization of a matrix (what it is and main result).

(b) If A has a Cholesky factorization and G is its Cholesky factor show that $\|G\|_2^2 = \|A\|_2$. Hint: use the SVD of G.

(c) Compute the Cholesky factorization of

$$\begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}.$$

(d) If $C(n)$ denotes the complexity (number of floating point operations) of the Cholesky factorization algorithm of a an $n \times n$ matrix A, show that $C(n) = n^3/3$. Hint: By using a block-form of the algorithm, i.e., writing the matrix A in the form

$$A = \begin{bmatrix} \alpha & u^T \\ u & B \end{bmatrix},$$

with u an $(n-1) \times 1$ matrix and B an $(n-1) \times (n-1)$ matrix (and a similar form for the Cholesky factor R), it is easy to derive a recursion formula between $C(n)$ and $C(n-1)$.

(4) (a) State sufficient conditions on a square matrix A so that the Jacobi iteration for solving the linear system

$$Ax = b \tag{1}$$

converges. How about the Gauss-Seidel iteration?

(b) With A as in (3 c), $b = [1, 1]^T$, and $x_0 = [0, 0]^T$, compute the iterates x_1, x_2 for both Jacobi and Gauss-Seidel iterations for the system (1).

(c) Compute the error propagators for both Jacobi and Gauss-Seidel for the linear system (1) with A chosen as in (3 c). Decide whether the iterations converge, and estimate the convergence rate (explain in what way/norm that convergence rate is relevant).