Math 620 Comprehensive Examination 8/21/2013

Name:

**Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.**

1. (a) Consider Newton's method to find the solution $\alpha$ of $f(x) = 0$, where $f(x), f'(x), f''(x)$ are all continuous, and $f'(x) \neq 0$ for any $x$. Suppose the iterates $\{x_n\}, n \geq 0$ converge to $\alpha$ for some choice of initial guess. Find the value of

$$\lim \frac{\alpha - x_{n+1}}{(\alpha - x_n)^2}.$$

   (b) Show that when $|\lambda| > 2$, the iteration

$$x_{n+1} = x_n + \lambda(x_n^2 + 3x_n + 2)$$

   will not converge to either root of

$$x^2 + 3x + 2 = 0$$

   for any initial guess (not equal to the root).

   (c) Newton's Method has order of convergence 2 and the Secant Method has order $\approx$ 1.62. However, this does not provide a complete picture of their relative efficiency. Explain briefly.

2. (a) Let $f(x)$ be a function satisfying

$$|f^{(i)}(x)| \leq 2^i i!, \; i = 1, \ldots n+1,$$

   and let $p(x)$ be its polynomial interpolant corresponding to the $x$-values $x_0, x_1, \ldots x_n$, all of which lie in $[0, \frac{1}{2}]$. Show that for any $x \in [0, \frac{1}{2}]$, we have

$$|f(x) - p(x)| \leq 1.$$

   Hint: Use the error formula for polynomial interpolation.

   (b) Let $n \geq 2$ and $x_1 = 1, x_2 = 2, \ldots, x_n = n$. Let $l_1(x), l_2(x), \ldots l_n(x)$ be the Lagrange polynomials corresponding to $\{x_i\}$. Show that

$$l_1(x) + 2l_2(x) + \cdots + nl_n(x) = x.$$

(c) The temperature $T$ is calculated every Friday at 10 AM for a full year at BWI airport to give a set of values $(i, T_i)$, where for $i = 1, 2, \dots 52$, $T_i$ denotes the temperature for week $i$. The polynomial interpolant satisfying $p(i) = T_i$, $i = 2, 4, \dots 52$ (even only) is constructed. For odd $i = 3, 5, \dots 49$, which do you expect to be a closer approximation to $T_i$: $p(i)$ or the average of $p(i-1)$ and $p(i+1)$? Discuss briefly.

3. (a) It is desired to find $a, b, c$ such that $r^*(x) = a + bx + cx^2$ minimizes

$$\int_0^1 x[x^7 - r(x)]^2\, dx$$

over all quadratics $r(x)$. Use direct minimization to derive a matrix equation $Az = F$ satisfied by $z = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. (Don't solve it.)

(b) Suppose $\{P_n\}$ are the Legendre polynomials, with the $L_2$ norm of $P_n$ given by $\sqrt{\frac{2}{2n+1}}$. Let $\{Q_n\}$ be orthonormal polynomials on $[-1, 1]$ corresponding to the weight $w(x) = x^2$. Compute the value of

$$I = \int_{-1}^{1} (P_4(x) + 3xQ_1(x) + xQ_2(x))^2\, dx.$$

Hint: Use orthonormality/orthogonality properties of both $\{P_n\}$ and $\{Q_n\}$ — don't try to explicitly find $Q_1, Q_2$.

4. Consider the Initial Value Problem (IVP)

$$y'(x) = f(x, y(x)) \quad \text{for } x_0 < x \le b, \quad y(x_0) = y_0,$$

with a given function $f(x, y)$ and initial condition $y_0$.

(a) Derive the explicit Euler method for the above IVP in three different ways.

(b) Discuss several important families of numerical methods for the IVP. It may be useful to explain how some of them can be thought of as generalizations of either the explicit or the implicit Euler method. Give the name of the method families and give some more detail, such as either state their general formula, or give some sample formulas, or discuss their typical advantages and disadvantages.