Math 630 Comprehensive Examination 8/21/2013

Name:

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.

1. (a) State at least two versions of the definition of the Singular Value Decomposition (SVD) of an $n \times m$ matrix A. [Hint: Think of full vs. reduced SVD.] Be precise in introducing all notation. For definiteness in sketching some of the notational connections, you may assume that $n > m$, i.e., that the matrix A has more rows than columns (like it has in an overdetermined least squares problem). Explain how the different versions of the SVD are connected; properly introduced notation might make this clear.

   (b) Is the SVD for a given matrix unique? In what sense? For certain non-uniqueness, you may want to give examples of small matrices, such as 2×2 matrices.

2. Consider a real square matrix $A \in \mathbb{R}^{n\times n}$.

   (a) State the definition for the matrix A to be positive definite.

   (b) Give at least three ways (in addition to the definition) to check if a given real square matrix is positive definite. One or more ways should be computational in nature, that is: if you had a matrix given in the computer, which algorithm might you run on it to determine computationally if it is positive definite?

   (c) Show that the matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ is positive definite.

3. (a) Let A be an $n \times n$ nonsingular matrix and $x, \delta x, b, \delta b \in \mathbb{R}^n$ with $x \neq 0$ be so that

$$Ax = b\ , \quad A(x + \delta x) = b + \delta b\ . \tag{1}$$

   Prove that

$$\frac{\|\delta x\|_\infty}{\|x\|_\infty} \leq \operatorname{cond}_\infty(A) \frac{\|\delta b\|_\infty}{\|b\|_\infty}\ . \tag{2}$$

   (b) Define the $n \times n$ matrix $A = I + \alpha e_1 e_n^T$ $(n > 1)$, where $e_i = [0, \ldots, 1, \ldots, 0]$ is the $n \times 1$ matrix representing the i^{th} vector of the standard basis in $\mathbb{R}^n$ and $\alpha \in \mathbb{R}$ is a parameter. Compute A^{-1}, $\sigma(A)$ (the spectrum of A), $\operatorname{cond}_\infty(A)$, and comment on the relationship between the spectrum of a matrix and its condition number. (Hint: A is unit upper triangular.)

(c) Let $b = e_1$ and $\delta b = \varepsilon e_n$, and consider the system (1) with A as defined at (b). Compute the quantities

$$\|\delta x\|_\infty \,/\, \|x\|_\infty \,, \quad \|\delta b\|_\infty \,/\, \|b\|_\infty \,,$$

and verify that they satisfy (2). (Note: do not just state that (2) is satisfied because you showed that it holds, verify that it holds specifically for this example.)

4. Given are two nonsingular $n \times n$ matrices A, M and the system

$$Ax = b \,, \tag{3}$$

with $b \in \mathbb{R}^n$ and solution $x^* = A^{-1}b$.

(a) Consider the simple iteration for finding x^*:

$$x_{n+1} = x_n + M^{-1}(b - Ax_n), \quad n = 0, 1, \ldots \,. \tag{4}$$

If $e_n = x^* - x_n$ is the error at step n, show that

$$\overset{x^* - x_{n+1}}{e_{n+1}} = (I - M^{-1}A)\overset{x^*}{e_n} \,.$$

(b) State (do not prove) a condition on A and M that is necessary and sufficient for x_n to converge to x^* regardless of the choice of b or x_0.

(c) With

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad \text{and} \quad M = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

prove that the iteration (4) converges for any $x_0, b \in \mathbb{R}^2$. Identify (by name) this iteration, and for $b = [1, 2]^T$, and $x_0 = [0, 0]^T$ find x_1, x_2, x_3.