Math 630 Comprehensive Examination 1/23/2014

Name:

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.

1. Let $\|\cdot\|$ denote a vector norm for $\mathbb{C}^n$, and let

$$\|A\| := \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

denote the associated operator norm.

   (a) State the general definition for $\|A\|$ to be a matrix norm for $A \in \mathbb{C}^{n\times n}$.

   (b) Show that $\|A\|$ is a matrix norm for $A \in \mathbb{C}^{n\times n}$.

   (c) Show that $\|Ax\| \leq \|A\| \, \|x\|$.

   (d) Show that the operator norm is consistent, that is, $\|AB\| \leq \|A\| \, \|B\|$.

2. Notice that the matrix

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}.$$

is real symmetric and thus it is guaranteed that there exists a diagonalization $A = UDU^T$ with an orthogonal transformation matrix U and a diagonal matrix D of eigenvalues. Compute the matrices U and D of the orthogonal diagonalization of A.

3. (a) Define the pseudo-inverse A^+ of a matrix $A \in \mathbb{R}^{m\times n}$ and briefly state its relationship with least-squares problems.

   (b) Show that for $A \in \mathbb{R}^{m\times n}$

$$\lim_{\varepsilon \to 0} (A^T A + \varepsilon I)^{-1} A^T = A^+ ,$$

   and if $rank(A) = n$, then

$$(A^T A)^{-1} A^T = A^+.$$

   (c) Compute A^+ where $A = [1, 2, 3]^T$.

4. Consider the problem of finding the solution $x \in \mathbb{R}^n$ of the system of linear equations $Ax = b$ with non-singular square matrix $A \in \mathbb{R}^{n \times n}$ and vector $b \in \mathbb{R}^n$.

   (a) Explain what the terms "direct method" and "iterative method" mean. Give some examples of methods that are direct or iterative methods, respectively. State the assumptions on the system matrix A that particular methods might require.

   (b) Discuss in which situations you would use direct and iterative methods. What are some advantages or disadvantages? Give examples of some situations in which you would prefer one method over the other.