Math 630 Comprehensive Examination 8/20/2014

Name:

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.

1. For a given vector $0 \neq v \in \mathbb{C}^n$, let $H = I - 2vv^*/(v^*v) \in \mathbb{C}^{n\times n}$ denote the Householder reflector.
   (a) Show that H is Hermitian.
   (b) Show that H is unitary.
   (c) Determine all eigenpairs of H.
   (d) What is the matrix-2-norm $\|H\|_2$ of H?
   (e) Compute the determinant of H.

2. Consider the overdetermined system of linear equations
$$Ax = b$$
with $A \in \mathbb{R}^{n\times m}$, $n > m$, $b \in \mathbb{R}^n$. The least-squares problem for this system consists of finding a vector $x \in \mathbb{R}^m$ such that the Euclidean vector norm of the residual vector $r = b - Ax$ is minimized.

   There are three well-known approaches to solve this problem. Describe at least two of them and discuss their relative advantages and disadvantages.

3. Let $A \in \mathbb{R}^{n\times n}$ be a non-singular matrix.
   (a) State (do not prove) a **necessary and sufficient** condition for A to have an LU-decomposition with L unit lower triangular.
   (b) If $A = LU$ is an LU-decomposition of A with L unit lower triangular, prove that the factors L and U are uniquely determined.
   (c) Let
$$A = \begin{bmatrix} \varepsilon & 1 \\ 1 & 1 \end{bmatrix},$$
   with $0 < \varepsilon < 1$. Compute the LU-decomposition of A (in exact arithmetic, L unit lower triangular). Assuming that ε is very small compared to 1 (to the point that $1+\varepsilon$ is numerically interpreted as equal to 1), explain the effect of round-off error on the numerical LU-factorization of A without pivoting.

4. Consider the iterative method

$$x_{n+1} = x_n + M^{-1}(b - Ax_n) \tag{1}$$

for solving the linear system $Ax = b$, where $A, M \in \mathbb{R}^{n \times n}$ are non-singular matrices, and denote by $x^* = A^{-1}b$ the solution.

(a) If $e_n = x^* - x_n$ is the error at the n^{th} step, show that

$$e_{n+1} = (I - M^{-1}A)e_n.$$

(b) State a **necessary and sufficient** condition on A and M in order for the sequence x_n resulting from the iteration (1) to converge to the solution x^* for any initial guess $x_0 \in \mathbb{R}^n$.

(c) Let

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix},$$

and $M = 2I$ (the matrix obtained by setting to zero all non-diagonal entries of A). Argue that the iteration (1) is convergent and give an estimate of the convergence rate (the quantity by which the size of the error is decreased at each step) in the appropriate norm.