Math 620 Comprehensive Examination 1/22/2015

Name:

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded. Note that each sub-problem is worth the same number of points.

1. (a) Describe (at least) three methods for the computation of an interpolating polynomial to a given dataset. Discuss their relative advantages, for instance, relative to computational cost or applicability or purpose of use. Hint: Start by clearly introducing notation for the problem to be solved.

   (b) Compute the interpolating polynomial to the data
$$\begin{array}{c|c|c|c} 0 & 4 & 2 & -2 \\ 1 & 2 & 0 & 2 \end{array}.$$
   Simplify your result to the form of a standard polynomial.

   (c) Suppose $p_n(x)$, a polynomial of degree n, interpolates the function $f(x)$ at the points $x_0 < x_1 < \ldots < x_n$. Let k be an integer satisfying $1 \leq k \leq n$. Prove the existence of $n+1-k$ points $\{y_k\}$ satisfying $x_0 < y_1 < y_2, \ldots < y_k < x_n$ where the kth derivative of f and p_n match (i.e. $f^{(k)}(y_i) = p_n^{(k)}(y_i)$).

2. (a) Consider the problem of finding the square root of a given real positive number a. Using the function definition $f(x) := x^2 - a$ in the root-finding problem, derive the recursive formula representing Newton's method for solving $f(x) = 0$.

   (b) Check the relevant assumptions, in particular those on $f(x)$, to apply the convergence theorem for Newton's method and conclude its second-order convergence for this problem (when the initial guess x_0 is chosen close enough to the desired answer). Apply the method to calculate four steps to compute $\sqrt{2}$ with initial guess $x_0 = 1$. How many decimal digits of accuracy does each iterate have? (The correct representation of $\sqrt{2}$, up to the first 15 digits, is 1.414213562373095.)

   (c) Suppose now that $b = -a$, where $a > 0$. Write the recursion representing Newton's method with $f_1(x) = x^2 - b$ now. Show that if the initial guess x_0 in the method in part (a) above leads to convergence to $\sqrt{a}$, then taking the initial guess $y_0 = ix_0$ for solving $f_1(x) = 0$ using Newton's method leads to convergence to $\sqrt{b} = i\sqrt{a}$.

3. (a) Let $I(f) = \int_a^b f(x)dx$. Let $T_n(f)$ be the approximation of $I(f)$ by the composite Trapezoidal rule with n equal subintervals (using their endpoints), $M_n(f)$ be the composite midpoint rule approximation (again using n subintervals, where now the midpoints are used) and $S_n(f)$ be the Simpson's rule approximation over the same n subintervals (where both endpoints and midpoints are used). Suppose $T_n(f) = 1.5$ and $S_n(f) = 1.6$. Find $M_n(f)$.

(b) Suppose now that f is a cubic function. Show that if $T_1(f)$ and $T_2(f)$ are given, then you can find the exact value of $I(f)$.

(c) Now let $a = -1, b = 1$ and let $Q(f)$ denote the k point Gauss-Legendre quadrature approximation to $I(f)$. Define $f(x) = P_k(x) + x^{k-1}$, where $\{P_n(x)\}$ denotes the orthonormal Legendre polynomials on $[-1, 1]$. Find the value of $I(f^2) - Q(f^2)$.

4. Consider the following two-step method for solving the initial value problem

$$y'(x) = f(x, y(x)), \quad y(0) = y_0$$

on the interval $[0, 1]$ using the uniform grid $x_k = hk$, $k = 0, 1, \dots, N$:

$$Y_{n+1} = \frac{3}{2}Y_n - \frac{1}{2}Y_{n-1} + h\left(\frac{5}{4}f(x_n, Y_n) - \frac{3}{4}f(x_{n-1}, Y_{n-1})\right), \tag{1}$$

where Y_n is supposed to approximate the value $y(x_n)$ for $0 \le n \le N$, and $h = 1/N$.

(a) Show that this is a second order method and compute the leading term in the truncation error.

(b) Determine whether the method (1) is convergent or not; if convergent, express the error as $O(h^p)$ for a certain power p, under appropriate assumptions.

(c) Implementing (1) requires finding a value Y_1 approximating $y(x_1)$. Explain how Y_1 can be found so that the order of the error at (b) is not affected.