Math 630 Comprehensive Examination 1/22/2015

Name:

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.

1. (a) State the theorem of the singular value decomposition with its necessary assumptions.

   (b) Compute the singular decomposition of the matrix $A = \begin{bmatrix} -1 & -1 \\ 2 & -2 \\ -1 & -1 \\ 2 & -2 \end{bmatrix}$.

2. (a) Define the pseudoinverse A^+ of a matrix $A \in M_{m \times n}$, and briefly (no arguments needed) explain its relationship with the least-squares problem

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|^2 , \tag{1}$$

   where $b \in \mathbb{R}^m$ is given and $\|\cdot\|$ denotes the 2−norm in $\mathbb{R}^m$.

   (b) Show that if $A \in M_{m \times n}$ has rank m (so $m \leq n$), then

$$A^+ = A^T(AA^T)^{-1} .$$

   (c) Solve the minimum-norm least-squares problem (1) for

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

3. (a) Let $A \in M_n$ be a square matrix. Starting from the definition of the matrix 2-norm

$$\|A\|_2 = \max_{\|x\|_2 = 1} \|Ax\|_2 ,$$

   show that for any orthogonal matrix $U \in M_n$ we have

$$\|UA\|_2 = \|AU\|_2 = \|A\|_2 ,$$

   and that the 2-norm of A is equal to the largest singular value of A.

   (b) If $A \in M_n$ is symmetric and positive definite and R is its Cholesky factor, show that

$$\|A\|_2 = \|R\|_2^2, \text{ and, } \operatorname{cond}_2(A) = \operatorname{cond}_2(R)^2 .$$

   Hint: Start from the singular value decomposition of R.

4. Consider the matrix

$$A = \begin{bmatrix} 9 & 1 \\ 1 & 2 \end{bmatrix}.$$

(a) Compute the eigenvalues of A and the iterates q_1, q_2 derived from the power method with $q_0 = [1, 0]^T$ (rescale q_j so that its first component is 1). What is the convergence rate of $\|q_n - v_1\|$ where v_1 is the eigenvector of A corresponding to the largest eigenvalue (rescaled so that its first component is 1)?

(b) Since 9 is a reasonable approximation of the largest eigenvalue of A, perform one step (compute q_1) of the shift-and-inverse iteration for A with $\rho = 9$ and $q_0 = [1, 0]^T$. What is the convergence rate of of $\|q_n - v_1\|$ for this iteration? Compare this rate with the rate obtained at (a).