Math 630 Comprehensive Examination 1/21/2016

Name:

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.

1. Let $A \in \mathbb{R}^{n\times n}$ be a matrix partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{11} \in \mathbb{R}^{k\times k}$ with $0 < k < n$.

(a) Show that there are unique matrices M and $\widetilde{A}_{22}$, of appropriate sizes, so that

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I_k & 0 \\ M & I_{n-k} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & \widetilde{A}_{22} \end{bmatrix}.$$

Prove that $\widetilde{A}_{22}$ is nonsingular if and only if A is nonsingular.

(b) Assume A is nonsingular and that it has a unit LU decomposition (L has a unit diagonal) partitioned as

$$A = \begin{bmatrix} L_1 & 0 \\ C & L_2 \end{bmatrix} \begin{bmatrix} U_1 & D \\ 0 & U_2 \end{bmatrix}.$$

Show that L_1U_1 and L_2U_2 are the LU decompositions of A_{11} and $\widetilde{A}_{22}$, respectively.

2. Let $A \in \mathbb{R}^{n\times m}$, and denote by $A^\dagger$ its pseudo-inverse of A.

(a) Briefly describe the construction of $A^\dagger$.

(b) Show that if $n \geq m$, and $\text{rank}(A) = m$, then $A^\dagger = \left(A^TA\right)^{-1}A^T$. Compare this result with normal equations $A^TAx = A^Tb$.

(c) Show that if $n \leq m$, and $\text{rank}(A) = n$, then $A^\dagger = A^T\left(AA^T\right)^{-1}$.

3. Let $\|\cdot\|$ be the matrix-norm on $\mathbb{R}^{n\times n}$ induced by a certain vector-norm $\|\cdot\|$ on $\mathbb{R}^n$.

(a) Let $B \in \mathbb{R}^{n\times n}$ be so that $\|B\| \le \varepsilon < 1$. Show that

$$\|(I-B)^{-1} - I\| \le \frac{\varepsilon}{1-\varepsilon} .$$

(Hint: use the series expansion of $(I-B)^{-1}$.)

(b) Let $A, E \in \mathbb{R}^{n\times n}$ so that A is nonsingular and $\|E\| \le \delta \|A\|$ with

$$\varepsilon = \mathrm{cond}(A)\,\delta < 1 ,$$

where $\mathrm{cond}(A) = \|A\| \cdot \|A^{-1}\|$. Furthermore, let $x, \widehat{x}, b \in \mathbb{R}^n$ be so that

$$Ax = b, \quad (A+E)\widehat{x} = b .$$

Show that

$$\frac{\|x-\widehat{x}\|}{\|x\|} \le \frac{\varepsilon}{1-\varepsilon} .$$

(Hint: Start by left-factoring out A in the equation $(A+E)\widehat{x} = b$.)

4. Consider the matrix $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.

(a) Starting with initial guess $q_0 = [1,1]^T$, compute two steps of the power method to find eigenvector approximations q_1 and q_2. Use the Rayleigh quotient to compute an approximation $\lambda_1^{(\text{power})}$ of the dominant eigenvalue λ_1 based on q_2.

(b) Also starting with $q_0 = [1,1]^T$, compute two steps of the inverse iteration method with shift $\rho = 9$. What motivates this choice of the shift? Use again the Rayleigh quotient to compute an approximation $\lambda_1^{(\text{inverse})}$ of the eigenvalue λ_1 based on q_2.

(c) Compute the true value of λ_1 and compare the quality of both approximations to the true value.