Math 620 Comprehensive Examination 8/23/2017

Name:

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.

1. This problem is concerned with solving the equation $f(x) = 0$ using Newton's method, assuming f is a smooth (C^∞) function.

   (a) Write the iteration representing Newton's method for solving $f(x) = 0$ and **briefly** state under what conditions the iteration **makes sense**.

   (b) Write Newton's method for solving the equation
   $$x^m = 0,$$
   where $m \geq 2$ is an integer. Show that the convergence is linear, not quadratic, and that
   $$\lim_{n\to\infty} \frac{e_{n+1}}{e_n} = \frac{m-1}{m}.$$

   (c) Consider the modified Newton iteration
   $$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}$$
   for finding roots with multiplicity $m \geq 2$; that is, it is assumed that
   $$f(x) = (x-a)^m g(x),$$
   with g being a smooth function so that $g(a) \neq 0$. Show that the iteration converges at least (locally) quadratically to the root a of f.
   Hint: It is helpful to realize that
   $$f'(x) = m(x-a)^{m-1}g_1(x), \text{ and } f''(x) = m(m-1)(x-a)^{m-2}g_2(x),$$
   with g_1, g_2 being smooth functions satisfying $g_1(a) = g_2(a) = g(a)$.

2. Consider the **non-standard** interpolation problem:
   Given three numbers A, B, C and a fixed node $\xi \in \mathbb{R}$, find a quadratic polynomial $q(x) = a_0 + a_1 x + a_2 x^2$ so that
   $$q(-1) = A, \quad q'(\xi) = B, \quad q(1) = C. \tag{1}$$

(a) Show that if $\xi = 0$, the problem is not well-posed, that is, there are numbers A, B, C for which the interpolation problem (1) has no solution.

(b) Show that if $\xi \neq 0$, then the interpolation problem (1) has a unique solution for any triplet A, B, C. (No need to establish an explicit formula for q).

(c) Assume $\xi \neq 0$ and $f \in C^3(\mathbb{R})$, and let

$$\Psi(x) = (x^2 - 1)(x + b), \quad \text{with } \; b = (1 - 3\xi^2)/(2\xi)$$

be the cubic polynomial satisfying $\Psi'(\xi) = 0$, $\Psi(-1) = \Psi(1) = 0$.
Show that the interpolation polynomial q found at (b) with $A = f(-1)$, $B = f'(\xi)$, and $C = f(1)$ satisfies the following error formula:
$\forall x \in \mathbb{R}$ there exists $\zeta \in \mathbb{R}$ so that

$$f(x) - q(x) = \frac{f^{[3]}(\zeta)}{3!}\Psi(x).$$

Hint: For a fixed $x \in \mathbb{R} \setminus \{-1, 1\}$ consider the function

$$r(t) = f(t) - q(t) - \Psi(t)\frac{f(x) - q(x)}{\Psi(x)}.$$

3. Consider the trapezoidal rule (T) and Simpson's rule (S) for approximating the integral of a relatively smooth function f on an interval $[a, b]$, for which the following error local estimates are known to hold:

$$\int_a^b f(x)dx - T(f) = -\frac{(b-a)^3}{12}f''(\eta), \quad \text{for some } \eta \in [a, b],$$

$$\int_a^b f(x)dx - S(f) = -\frac{\delta^5}{90}f^{[4]}(\zeta), \quad \text{for some } \zeta \in [a, b],$$

where $\delta = (b - a)/2$.

(a) Given a uniform partition $a = x_0 < x_1 < \cdots < x_n = b$ of an interval $[a, b]$ with an **even number** n of intervals, derive formulas for the composite trapezoidal and Simpson's rules both based on the values of the function f at **the same** nodes, i.e., $f(x_0), f(x_1), \ldots, f(x_n)$.

(b) Use the local estimates provided to derive error estimates for the composite rules, in terms of $h = x_{i+1} - x_i$ and certain quantities related to the relevant derivatives of f.

(c) Consider using the composite rules at (a) for aproximating $\int_0^{10} \sin(\pi x)dx$. Decide, based on the number of intervals n used and your estimates at (b), which of the composite rule is expected to produce a better approximation of the integral.

4. Given is the following multistep method for solving the ordinary differential equation $y'(x) = f(x, y(x))$ with uniform time step h:

$$y_{n+1} = y_n + \frac{h}{2}\left(3f(x_n, y_n) - f(x_{n-1}, y_{n-1})\right) \tag{2}$$

(a) Apply the method to compute $y_n \approx y(x_n)$ $(n = 1, 2, 3)$ for the initial value problem

$$y'(x) = -a\, y(x), \quad y(0) = 1,$$

where $x_j = h \cdot j$ and $h = 1/n$. What choices do you have to make when applying (2)?

(b) Decide whether the scheme (2) is consistent, stable, or convergent. Determine the order of convergence. If convergent, how should y_1 be computed in order to not affect the order of convergence?